 J. GUIDANCE

Appendix

By expanding the A matrix with the B matrix as its first column, the following approximate expressions for the zeros may be derived (neglecting unimportant contributions)

$$\omega_{n_1}^2 \approx [V_0/\mu\bar{c})]^2(\bar{c}/i_y)^2[\mu C_{m_\alpha} + C_{L_\alpha}(C_{m_q} + C_{m_{\dot\alpha}})/2]$$

$$\sigma_1 \approx V_0/(2\mu\bar{c})[C_{L_\alpha} - (\bar{c}/i_y)^2(C_{m_q} + C_{m_{\dot\alpha}})/2] \qquad \text{(A1)}$$

$$\omega_{n2}^2 \approx -C_L V_0^2 \rho_h/(\mu\bar{c}) = -g\rho_h$$

$$\sigma_2 \approx 0 \qquad \text{(A2)}$$

Equations (A1) agree with the well-known approximations for the short-period mode (σ_α, ω_{n_α}), thus resulting in Eqs. (2). With regard to Eqs. (A2), the following phugoid approximations resulting from expanding the A matrix may be considered

$$\omega_{n_p}^2 \approx -g\rho_h + \left(2 + \frac{C_{L_u}}{2C_L}\right)\left(\frac{g}{V_0}\right)^2 = -g\rho_h\left[1 - \left(2 + \frac{C_{L_u}}{2C_L}\right)k_\rho\right]$$

$$\sigma_p \approx \left(1 + \frac{C_{L_u}}{2C_L}\right) k_\rho n_u \frac{g}{V_0}\frac{C_D}{C_L} \qquad \text{(A3)}$$

Combining these expressions with Eqs. (A2), the correlation described by Eqs. (2) results.

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Generalized Gradient Algorithm for Trajectory Optimization

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Introduction

EQUALITY constraints represent a general class of path constraints in trajectory optimization,[1] since many types of constraints can be converted into path equality constraints. Miele[1] has developed an algorithm named Sequential Gradient Restoration Algorithm (SGRA),[1] which can solve general trajectory optimization problems. Goh and Teo[2] proposed a unified control parameterization approach that converts an optimal control problem into a parameter optimization problem. These algorithms employ a sequence of cycles, each cycle having two phases. One phase improves the performance index while the other reduces the constraint violations.

In this Note, a generalized gradient is found that improves the performance index and reduces the constraints at the same time.

Problem Statement

The general problem that will be considered has the following form:

$$\min_{u,\pi} I = \phi[x(1),\pi] + \int_0^1 L(x, u, \pi, \tau)\, d\tau \qquad (1)$$

subject to

$$\dot{x} = f(x, u, \pi, \tau) \qquad (2)$$

and $x(0)$ is given with the following constraints:

$$\psi[x(1),\pi] = 0 \qquad (3)$$

$$S(x, u, \pi, \tau) = 0 \qquad (4)$$

where u is the $m \times 1$ control vector, x is the $n \times 1$ state vector, π is the $p \times 1$ parameter vector, τ is the normalized time in (0,1), ψ is the $q \times 1$ terminal constraint vector, and S is the $r \times 1$ path equality constraint vector. A meaningful problem requires: $r \le m$ and $q \le n + p^* \le n + p$, where p^* is the number of parameters in ψ.[1]

Let us first consider the problem without terminal constraints. If there is no path equality constraint, the problem can be handled in the conventional way:

$$J = \phi + \int_0^1 \left[L + \lambda^T(f - \dot{x})\right] d\tau$$

$$= (\phi - \lambda^T x)_1 + (\lambda^T x)_0 + \int_0^1 (L + \lambda^T f + \dot{\lambda}^T x)\, d\tau$$

Define

$$H \triangleq L + \lambda^T f$$

The first variation of the augmented cost functional is then

$$\delta J = (\phi_x - \lambda^T)_1 \delta x_1 + \left[(\phi_\pi)_1 + \int_0^1 H_\pi\, d\tau\right] \delta\pi$$

$$+ \int_0^1 \left[(H_x + \dot{\lambda}^T)\delta x + H_u \delta u\right] d\tau \qquad (5)$$

When path constraints exist, the following r equations must be satisfied to maintain the path constraints to first order:

$$S_x \delta x + S_u \delta u + S_\pi \delta\pi = 0 \qquad (6)$$

If the path equality constraints contain control components, $|S_u S_u^T| \neq 0$ for $0 \le \tau \le 1$. The general solution of the algebraic equation, Eq. (6), is

$$\delta u = -D_u S_x \delta x - D_u S_\pi \delta\pi + \delta^* u \qquad (7)$$

where $\delta^* u$ is the homogeneous part: $S_u \delta^* u = 0$ and $D_u \triangleq S_u^T (S_u S_u^T)^{-1}$.

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Substituting Eq. (7) into Eq. (5), one obtains

$$\delta J = (\phi_x - \lambda^T)_1 \delta x_1 + \left[(\phi_\pi)_1 + \int_0^1 (H_\pi - H_u D_u S_\pi)\, d\tau\right]\delta\pi + \int_0^1 \left[(H_x + \dot\lambda^T - H_u D_u S_x)\delta x + H_u \delta^* u\right] d\tau \tag{8}$$

Choose

$$\dot\lambda^T = -H_x + H_u D_u S_x \quad \text{and} \quad (\lambda^T)_1 = (\phi_x)_1 \tag{9}$$

$$\delta\pi = -\left[(\phi_\pi)_1 + \int_0^1 (H_\pi - H_u D_u S_\pi)\, d\tau\right]^T \tag{10}$$

Then we would like to choose $\delta^* u$ such that $H_u \delta^* u < 0$ and $S_u \delta^* u = 0$. The use of the Lagrange multiplier method gives

$$\delta^* u = -[H_u - H_u D_u S_u]^T \tag{11}$$

The differential equations are satisfied to first order:

$$\delta\dot x = f_x \delta x + f_u \delta u + f_\pi \delta\pi \tag{12}$$

where δx can be obtained by forward integrating Eq. (12) with zero initial conditions and substitutions from Eqs. (10) and (11) and the "feedback" relation, Eq. (7). Then δu is found from Eq. (7), and changes may be made as $\Delta\pi = K\delta\pi$ and $\Delta u = K\delta u$, where K is the stepsize.

It is also possible to reduce path constraint violations at the same time. This requires choosing δu and $\delta\pi$ to make the path constraint variations: $dS = -\rho S_0$, where $\rho > 0$, S_0 is the path constraint vector evaluated at the current step, and $dS = S_x \delta x + S_u \delta u + S_\pi \delta\pi$.

Therefore

$$S_x \delta x + S_u \delta u + S_\pi \delta\pi + \rho S_0 = 0 \tag{13}$$

The same arguments are carried out as before until selection of $\delta^* u$, which has to satisfy $H_u \delta^* u < 0$ and $S_u \delta^* u + \rho S_0 = 0$. The use of the Lagrange multiplier method gives

$$\delta^* u = -(I - D_u S_u) H_u^T - \rho D_u S_0 \tag{14}$$

where $\delta^* u$ consists of two parts. The first term improves the performance index while maintaining the path constraints to the first order. The second term reduces the path constraints.

The impulse response method,[3,4] is employed to seek the solution of the problem when terminal constraints are also present. Define the following quantities:

$$\psi_0 = \phi$$

$$H^{(0)}(\tau) = L(\tau) + [\lambda^{(0)}]^T f(\tau)$$

$$H^{(j)}(\tau) = [\lambda^{(j)}]^T f(\tau) \qquad j = 1,\ldots, q$$

$$P_\pi^{(j)} = (\psi_{j\pi})_1 + \int_0^1 (H_\pi^{(j)} - H_u^{(j)} D_u S_\pi)\, d\tau \qquad j = 0,\ldots, q$$

Choose for $j = 0, 1,\ldots, q$

$$\dot\lambda^{(j)} = -[H_x^{(j)} - H_u^{(j)} D_u S_x]^T \quad \text{and} \quad (\lambda^{(j)})_1 = (\psi_{jx})_1^T$$

From Eq. (8), we have

$$\delta J = \int_0^1 H_u^{(0)} \delta^* u\, d\tau + P_\pi^{(0)} \delta\pi \tag{15}$$

$$\delta\psi_j = \int_0^1 H_u^{(j)} \delta^* u\, d\tau + P_\pi^{(j)} \delta\pi \tag{16}$$

where $j = 1,\ldots, q$, together with the requirement: $S_u \delta^* u + \rho S_0 = 0$.

Now, adjoin Eq. (16) to Eq. (15) with q Lagrange multipliers μ_j:

$$\delta\bar J = \delta J + \sum_{j=1}^q \mu_j \delta\psi_j = \int_0^1 H_u(\tau)\delta^* u\, d\tau + P_\pi \delta\pi \tag{17}$$

where

$$H_u(\tau) = H_u^{(0)}(\tau) + \sum_{j=1}^q \mu_j H_u^{(j)}(\tau)$$

$$P_\pi = P_\pi^{(0)} + \sum_{j=1}^q \mu_j P_\pi^{(j)}$$

According to the preceding discussions:

$$\Delta\pi = -K P_\pi^T$$

$$\Delta^* u = -K(I - D_u S_u) H_u^T - K\rho D_u S_0$$

The predicted terminal constraint variations for $j = 1,\ldots, q$ are:

$$\Delta\psi_j = \int_0^1 H_u^{(j)} \Delta^* u\, d\tau + P_\pi^{(j)} \Delta\pi \triangleq -K\left(g_j + \sum_{i=1}^q \mu_i Q_{ij}\right)$$

where

$$g_j = \int_0^1 [H_u^{(j)}](I - D_u S_u)[H_u^{(0)}]^T d\tau + [P_\pi^{(j)}][P_\pi^{(0)}]^T + \rho\int_0^1 [H_u^{(j)}] D_u S_0\, d\tau$$

$$Q_{ij} = \int_0^1 [H_u^{(j)}](I - D_u S_u)[H_u^{(i)}]^T d\tau + [P_\pi^{(j)}][P_\pi^{(i)}]^T$$

In vector form, we have:

$$-\frac{\Delta\psi}{K} = g + Q\mu$$

If we choose $\Delta\psi = -\epsilon\psi$ where $0 < \epsilon \le 1$, the Lagrange multiplier can be evaluated as

$$\mu = Q^{-1}\left(\frac{\epsilon}{K}\psi - g\right) \tag{18}$$

The existence of Q^{-1} is one necessary condition for the algorithm to converge (the controllability condition). The other is that the path constraints must contain control components.

Algorithm

The algorithm is conveniently divided into two phases. The *feasibility phase* is designed to obtain a solution satisfying both the path constraints and the terminal constraints:

$$\min I_F = \frac{1}{2}(\psi^T\psi) + \frac{1}{2}\int_0^1 S^T S\, d\tau \tag{19}$$

subject to: $\dot x = f(x, u, \pi, \tau)$ with $x(0)$ given.

Usually, constraints represent hard physical conditions to be satisfied. If this problem does not converge, relaxation of the path constraints and/or terminal is necessary.

The *optimization phase* uses the results of the feasibility phase as the initial guesses. The variations in control and parameter vectors are made to improve the performance index as well as to reduce the constraint violations.

For the convenience of programming, the following quantities are defined:

$$D_u \triangleq S_u^T(S_uS_u^T)^{-1} \qquad (m \times r)$$
$$\bar{S}_x \triangleq D_uS_x \qquad (m \times n)$$
$$\bar{S}_\pi \triangleq D_uS_\pi \qquad (m \times p)$$
$$\bar{S}_0 \triangleq D_uS_0 \qquad (m \times 1)$$
$$I_m \triangleq I - D_uS_u \qquad (m \times m)$$
$$\bar{L}_x \triangleq L_x - L_uD_uS_x \qquad (1 \times n)$$
$$\bar{L}_\pi \triangleq L_\pi - L_uD_uS_\pi \qquad (1 \times p)$$
$$\bar{f}_x \triangleq f_x - f_uD_uS_x \qquad (n \times n)$$
$$\bar{f}_\pi \triangleq f_\pi - f_uD_uS_\pi \qquad (n \times p)$$

where $I_m^2 = I_m$ and $I_m^T = I_m$. Now, the algorithm can be stated step by step as follows:

1) Guess the initial control and parameter vectors: $u(\tau)$, π.

2) Integrate forward: $\dot{x} = f(x, u, \pi, \tau)$ with $x(0)$ given.

3) Compute the gradients: $S_x, S_u, S_\pi, f_x, f_u, f_\pi, \psi_x, \psi_\pi$, and I_F. If $I_F < \epsilon_f$, go to step 8, where ϵ_f is a preselected small number, e.g., 10^{-4}. Otherwise, continue.

4) Integrate backward: $\dot{\lambda} = -S_x^TS - f_x^T\lambda^T$ with $(\lambda)_1 = (\psi_x^T\psi)_1$.

5) Compute the feasibility phase gradients:

$$h_u = S^TS_u + \lambda^Tf_u$$

$$p_\pi = (\psi^T\psi_\pi)_1 + \int_0^1 (S^TS_\pi + \lambda^Tf_\pi)\, d\tau$$

6) Improve controls and parameters. With the steepest descent method, $\Delta_u = -K_Fh_u^T$ and $\Delta\pi = -K_Fp_\pi^T$. Go to step 2.

7) Integrate forward: $\dot{x} = f(x, u, \pi, \tau)$ with $x(0)$ given.

8) Compute the following generalized gradients:

$$S_0\ \bar{S}_0\ \psi\ \bar{S}_x\ \bar{S}_\pi\ I_m\ \bar{L}_x\ L_u\ \bar{L}_\pi$$

$$\bar{f}_x\ f_u\ \bar{f}_\pi\ \psi_{jx}\ \psi_{j\pi} \quad \text{for} \quad j = 0,1,\ldots, q \quad \text{and} \quad I_F$$

9) Perform $q+1$ backward integrations, where $j = 1,\ldots, q$.

$$\dot{\lambda}^{(0)} = -\bar{L}_x^T - \bar{f}_x^T\lambda^{(0)} \quad \text{with} \quad [\lambda^{(0)}{}_1] = (\psi_{0x})_1^T$$

$$\dot{\lambda}^{(j)} = -\bar{f}_x^T\lambda^{(j)} \quad \text{with} \quad [\lambda^{(j)}{}_1] = (\psi_{jx})_1^T$$

10) Compute the gradients with respect to controls:

$$H_u^{(0)} = L_u + [\lambda^{(0)}]^Tf_u$$

$$H_u^{(j)} = [\lambda^{(j)}]^Tf_u \qquad j = 1,\ldots, q$$

and the gradients with respect to parameters:

$$P_\pi^{(0)} = (\psi_{0\pi})_1 + \int_0^1 \left\{\bar{L}_\pi + [\lambda^{(0)}]^T\bar{f}_\pi\right\} d\tau$$

$$P_\pi^{(j)} = (\psi_{j\pi})_1 + \int_0^1 [\lambda^{(j)}]^T\bar{f}_\pi\, d\tau \qquad j = 1,\ldots, q$$

Then, choose ρ and compute the generalized gradients with respect to controls:

$$\bar{H}_u^{(0)} = H_u^{(0)}I_m + \rho\bar{S}_0 \quad \text{and} \quad \bar{H}_u^{(j)} = H_u^{(j)}I_m$$

11) Choose K and ϵ. Compute the following for $j = 1,\ldots, q$ and then μ using Eq. (18).

$$g_j = \int_0^1 [H_u^{(j)}][\bar{H}_u^{(0)}]^T\, d\tau + [P_\pi^{(j)}][P_\pi^{(0)}]^T$$

$$Q_{ij} = \int_0^1 [\bar{H}_u^{(j)}][\bar{H}_u^{(i)}]^T\, d\tau + [P_\pi^{(j)}][P_\pi^{(i)}]^T$$

12) Compute the generalized gradients:

$$\bar{H}_u(\tau) = \bar{H}_u^{(0)}(\tau) + \sum_{j=1}^{q} \mu_j\bar{H}_u^{(j)}(\tau)$$

$$P_\pi = P_\pi^{(0)} + \sum_{j=1}^{q} \mu_jP_\pi^{(j)}$$

and check the following stopping criteria:

$$\int_0^1 \bar{H}_u\bar{H}_u^T\, d\tau \le \epsilon_0 \quad \text{and} \quad P_\pi P_\pi^T \le \epsilon_0$$

where ϵ_0 is a preselected small number as ϵ_f, e.g., 10^{-4}. If the preceding conditions are satisfied and $I_F < \epsilon_f$, the algorithm has converged. If the preceding conditions are satisfied but $I_F > \epsilon_f$, go to step 3. Otherwise, go on.

13) Integrate forward: $\delta\dot{x} = \bar{f}_x\delta x - \bar{f}_\pi P_\pi^T - f_u\bar{H}_u^T$ with $(\delta x)_0 = 0$. Compute the improving control variations: $\delta u = -\bar{S}_x\delta x + \bar{S}_\pi P_\pi^T - \bar{H}_u^T$.

14) Improve controls and parameters: $\Delta\pi = -KP_\pi^T$ and $\Delta u = K\delta u$. Go to step 7.

The program also should be terminated when the number of cycles exceeds a specified value.

Comments

User's inputs include the feasibility accuracy ϵ_f, the optimal accuracy ϵ_0, the stepsize K, the terminal constraint improving factor ϵ, and the path constraint improving factor ρ.

The optimality phase takes $q+3$ integrations of nth order equation at each iteration. This algorithm uses the forward integrations of the system equation and the perturbation equation in feedback form and the backward integration of the adjoint equations. These integrations are numerically stable. In addition, the optimality phase reduces the constraint violations while optimizing the performance. As a result, the feasibility phase only needs to be inserted once with a proper ρ.

Example with Bounded Control

The example is a minimum-time problem for a double-integrator plant with bounded control[3]

$$\min_u I = t_f$$

subject to

$$\dot{x}_1 = t_fx_2, \qquad \dot{x}_2 = t_fu$$

The initial conditions are:

$$x_1(0) = 0, \qquad x_2(0) = 0$$

The terminal constraints are:

$$x_1(1) = 1, \qquad x_2(1) = 0$$

The control is bounded as $-1 \le u \le 1$.

Solution of this optimization problem involves "bang-bang" control.[3]

The control inequality constraint can be converted into a path equality constraint[5] by introducing an auxiliary control u_2 and denoting the original control as u_1:

$$u_1^2 + u_2^2 - 1 = 0$$

We now have the following equivalent problem:

$$\min_{u_1,u_2} I = t_f$$

subject to the original system equations with the preceding initial conditions and terminal constraints and the path equality constraint.

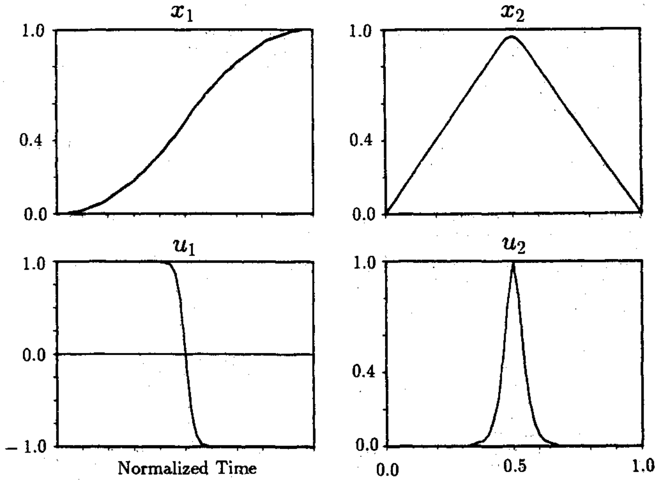

Fig. 1 Problem with bounded control.

The initial guesses of the controls and final time are:

$$u_1(\tau) = 1 \qquad u_2(\tau) = 1, \qquad t_f = 1$$

In the current implementation, the steepest descent scheme is employed in the feasibility phase. The program parameters are: $K_{uf} = 0.2$, $K_{pf} = 0.2$, $K_{uo} = 0.05$, $K_{po} = 0.05$, $\rho = 0.6$, $\epsilon = 0.1$, $\epsilon_f = 0.0001$, and $\epsilon_o = 0.001$, where K_{uf}, K_{pf} denote the stepsizes for control and parameter, respectively, in the feasibility phase, whereas K_{uo}, K_{po} are in the optimization phase.

The optimal minimum time is found to be $t_f^* = 2.0023$. This compares well with the analytical solution obtained in Ref. 3 ($t_f = 2.000$). The basic feature of a "bang-bang" control is clearly demonstrated. The results are plotted in Fig. 1.

The computer program is coded in C language. Details of the algorithm and more examples can be found in Ref. 6.

Conclusion

A generalized gradient algorithm has been proposed and verified for trajectory optimization problems with possible path equality constraints, terminal equality constraints, and control parameters. The generalized gradient was shown to improve the performance index and reduce the constraints at the same time.

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